MATH 2028 - Change of Variables Theorem

GOAL: Derive a general change of variables formula for multiple integrals

Recall: (Method of substitution)

If $g:[a,b]\to\mathbb{R}$ is C^1 and $f:\mathbb{R}\to\mathbb{R}$ is cts,

then
$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f\circ g(t)\cdot g'(t)\,dt$$

Reason: Let $F(x) := \int_0^x f(y)\,dy$. Then $F'(x) = f(x)$ by Fund. Thm. of Calculus. On the other hand.

Chain Rule $\Rightarrow$

$$(F\circ g)'(t) = F'(g(t))\cdot g'(t) = f\circ g(t)\cdot g'(t)$$

Integrate both sides from a to b and apply Fund. Thm. of Calculus again.

$$\begin{aligned}
\text{L.H.S.} &= \int_a^b (F\circ g)'(t)\,dt \\
&= (F\circ g)(b) - (F\circ g)(a) \\
&= \int_0^{g(b)} f(y)\,dy - \int_0^{g(a)} f(y)\,dy = \int_{g(a)}^{g(b)} f(y)\,dy
\end{aligned}$$

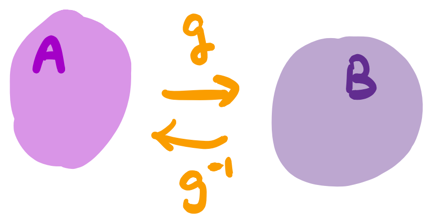
Suppose that $g : [a,b] \to \mathbb{R}$ is 1-1 and $g([a,b]) = [c,d]$. Then we have

$$\boxed{\int_c^d f(x)\,dx = \int_a^b f \circ g(t) \cdot |g'(t)|\,dt}$$

This is the Change of Variables Theorem in 1D.

<u>Q</u>: How to generalize this to higher dimensions?

Recall that a map $g : A \to B$ is said to be a (C^1)-diffeomorphism between the open subsets $A, B \subseteq \mathbb{R}^n$ if



- g is bijective
- both g and g^{-1} are C^1

<u>Remark</u>: By Inverse Function Theorem, a C^1 map $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image $g(A) = B$ provided that g is 1-1 and $\det(Dg) \neq 0$ everywhere.

Change of Variables Theorem

Let $g : A \to B$ be a diffeomorphism between two bounded open subsets $A, B \subseteq \mathbb{R}^n$ with measure zero boundary. For any cts $f : B \to \mathbb{R}$, we have

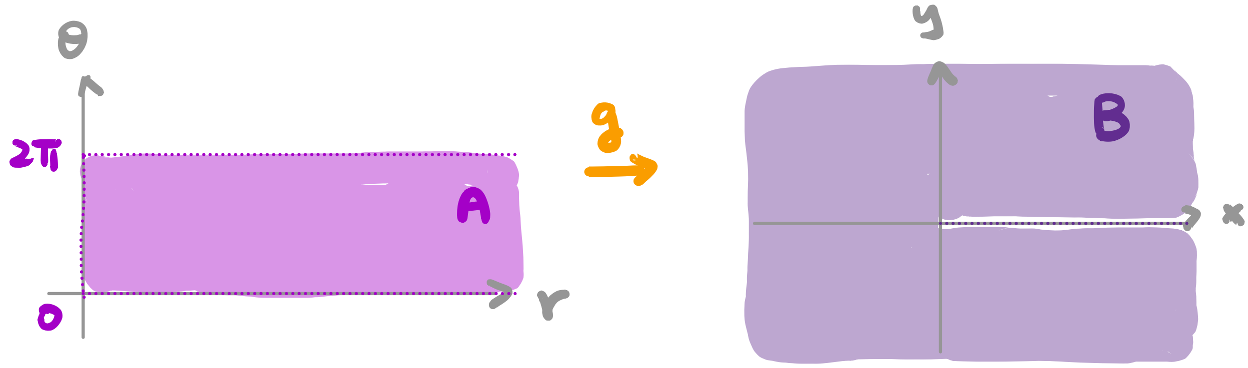
$$\boxed{\int_B f \, dV = \int_A (f \circ g) \cdot |\det(Dg)| \, dV} \qquad (*)$$

We will postpone the proof until later.

Let us verify formally how this change of variable formula yields the correct formula for the special coordinates discussed before.

Example 1: (Polar coordinates)

$$g : (0, \infty) \times (0, 2\pi) \longrightarrow \mathbb{R}^2 \setminus \{(x, 0) \mid x \geq 0\}$$



$$g(r, \theta) := (r\cos\theta, r\sin\theta)$$

Note that g is a bijective map from the infinite strip A onto the whole plane minus the non-negative x-axis B. Moreover,

$$Dg = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and $\det(Dg) = r > 0$ everywhere in A

Hence, $(*)$ implies the formula

$$dA = dx\,dy = r\,dr\,d\theta .$$

$\blacksquare$

Example 2: (Cylindrical coordinates)

$$g : (0,\infty) \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{(x,0,z) \mid x \geq 0\}$$

$$g(r,\theta,z) := (r\cos\theta, r\sin\theta, z) \qquad \text{bijective!}$$

$$Dg = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

& $\det(Dg) = r > 0$ Hence, $dV = dx\,dy\,dz$
$= r\,dr\,d\theta\,dz$.

$\blacksquare$

Example 3: (Spherical coordinates)

$$g : (0,\infty) \times (0,\pi) \times (0,2\pi) \to \{(x,0,z) \mid x \geq 0\}$$

$$g(\rho,\phi,\theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$$

$$Dg = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

& $\det(Dg) = \rho^2 \sin\phi > 0$

Hence, $dV = dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$.

$\square$

Sometimes we have to be a bit more careful to apply the change of variable formula.

Example 4: Evaluate $\int_D x^2 y^2 \, dA$ over the open disk Ω of radius 1 in $\mathbb{R}^2$ centered at the origin.

Solution: Note that we CANNOT cover the entire Ω with the polar coordinate system from an OPEN SUBSET. To do this properly, we first observe that $f(x,y) := x^2y^2$ is cts & bdd on Ω, hence f is also integrable on the open set $B := \Omega \setminus \{(x,y) \mid x \geq 0\}$ and

$$\int_\Omega f \, dA = \int_B f \, dA$$

since $\{(x,y) \mid x \geq 0\}$ has measure zero.

Now, B can be covered by the polar coord $g : (0,1) \times (0,2\pi) \to B$. Hence, by $(*)$,

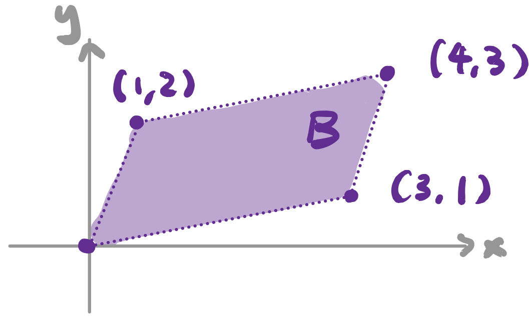
$$\int_B f \, dA = \int_0^{2\pi} \int_0^1 r^4 \sin^2\theta \cos^2\theta \cdot r \, dr \, d\theta$$

$$= \left(\int_0^1 r^5 \, dr \right) \cdot \left(\int_0^{2\pi} \sin^2\theta \cos^2\theta \, d\theta \right)$$

$$= \frac{\pi}{24}$$

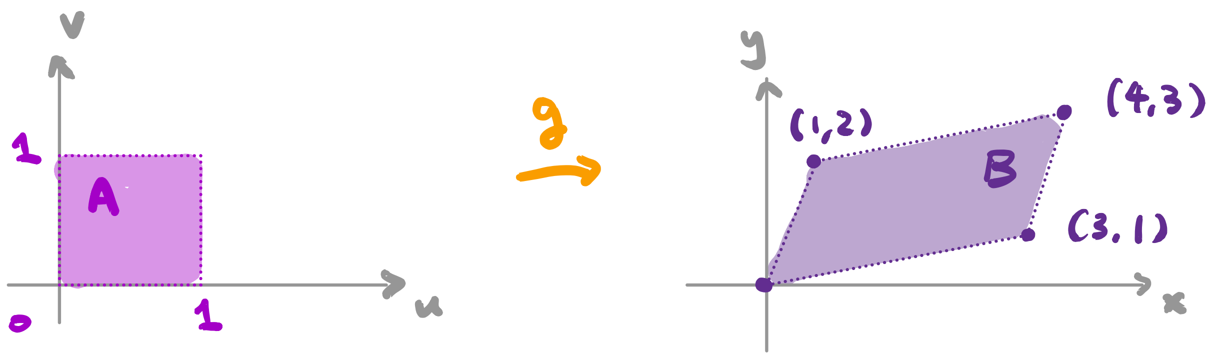
$\blacksquare$

Example 5: Evaluate $\int_B x \, dA$ where $B \subseteq \mathbb{R}^2$ is the open parallelogram



Solution: It is rather tedious to compute the integral in x, y coordinates. We can perform a linear change of variable first

$$g : (0,1) \times (0,1) \longrightarrow B$$



$$g(u,v) := (3u+v,\ u+2v)$$

Then, $Dg = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ & $\det(Dg) = 5 > 0$

Applying (*) gives

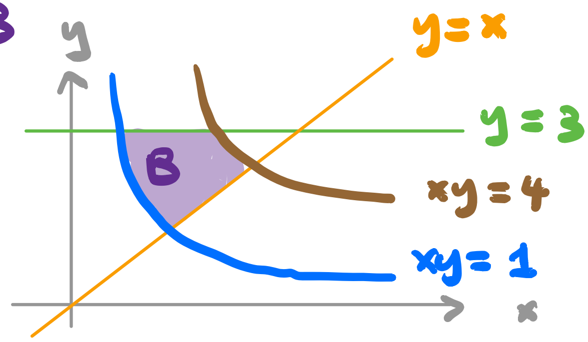
$$\int_B x \, dA = \int_0^1 \int_0^1 (3u+v) \cdot 5 \, du \, dv$$

$$= 5 \int_0^1 \left(\frac{3}{2} + v\right) dv = 5 \cdot \left(\frac{3}{2} + \frac{1}{2}\right) = 10$$

$\blacksquare$

Example 6: Evaluate $\int_B y \, dA$ where $B \subseteq \mathbb{R}^2$

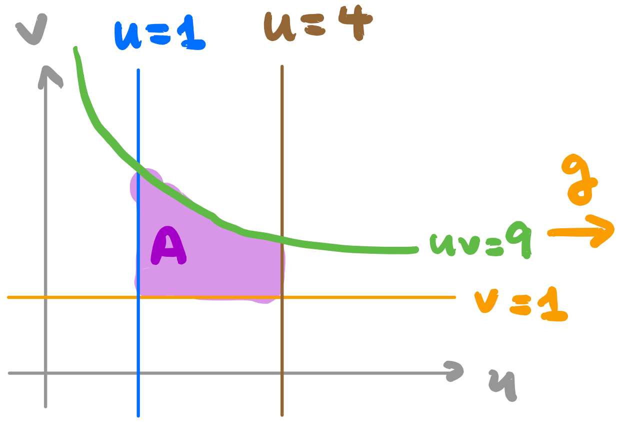
is the open subset

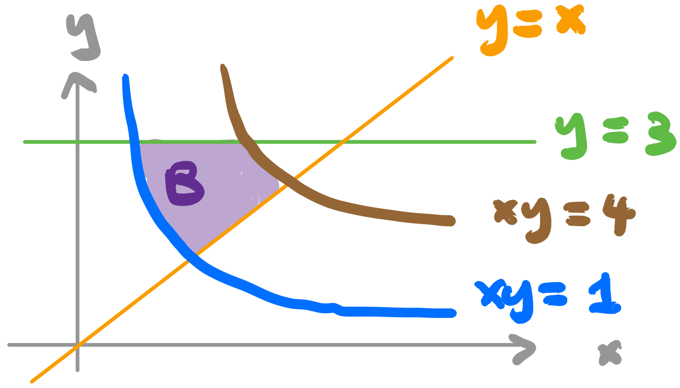


WANT:
$u = xy$
$v = y/x$

Solution: Define

$$g : A \longrightarrow B$$





$$g(u,v) := \left(\sqrt{\frac{u}{v}}, \sqrt{uv} \right)$$

$$Dg = \begin{pmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix}$$

& $\det(Dg) = \frac{1}{2v} > 0$ inside A

Apply (*), we obtain

$$\iint_B y\,dA = \int_1^4 \int_1^{9/u} \sqrt{uv} \cdot \frac{1}{2v}\,dv\,du$$

$$= \int_1^4 \frac{1}{2}\sqrt{u} \left(\int_1^{9/u} v^{-1/2}\,dv \right) du$$

$$= \int_1^4 \frac{1}{2}\sqrt{u} \left[2v^{1/2} \right]_1^{v=9/u} du$$

$$= \int_1^4 \sqrt{u} \left(\frac{3}{\sqrt{u}} - 1 \right) du$$

$$= \int_1^4 (3 - \sqrt{u})\,du$$

$$= \left[3u - \frac{2}{3}u^{3/2} \right]_{u=1}^{u=4} = \frac{13}{3}$$